Discussion on a small quadratic problem

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Question If $x^4 + \frac{1}{x^4} = 10$, find $x + \frac{1}{x}$.

1. Method 1

Let $y = x + \frac{1}{x}$, then squaring we get:

$$y^2 = \left(x + \frac{1}{x}\right)^2 = x^2 + 2x\left(\frac{1}{x}\right) + \frac{1}{x^2} = x^2 + 2 + \frac{1}{x^2} \ .$$

$$\therefore \ y^2 - 2 = x^2 + \frac{1}{x^2}$$

Squaring again, $y^4 - 4y^2 + 4 = x^4 + 2 + \frac{1}{x^4}$

$$y^4 - 4y^2 + 2 = x^4 + \frac{1}{x^4} = 10 \ , \quad \text{by given}$$

$$y^4 - 4y^2 - 8 = 0$$

$$y^2 = 2 \pm \sqrt{12}$$

$$\therefore \quad y = \pm\sqrt{2 + \sqrt{12}} \quad \text{or} \quad \pm i\sqrt{\sqrt{12} - 2}$$

$$\approx \pm 2.33754 \quad \text{or} \quad \pm 1.21\,\mathbf{i}$$

You may reject the complex roots $\pm\mathbf{i}\sqrt{\sqrt{12} - 2}$ if you work only on real numbers.

2. Method 2

Since $x^4 + \frac{1}{x^4} = 10$, $x^4 + 2 + \frac{1}{x^4} = 12$.

We get $\left(x^2 + \frac{1}{x^2}\right)^2 = 12$.

$$x^2 + \frac{1}{x^2} = \pm\sqrt{12}$$

$$x^2 + 2 + \frac{1}{x^2} = 2 \pm \sqrt{12}$$

$$\left(x + \frac{1}{x}\right)^2 = 2 \pm \sqrt{12}$$

$$x + \frac{1}{x} = \pm\sqrt{2 \pm \sqrt{12}} = \pm\sqrt{2 + \sqrt{12}} \quad \text{or} \quad \pm\mathbf{i}\sqrt{\sqrt{12} - 2}$$

To make you think more seriously on the problem, I propose two methods below.

Do you think that **(1)** the methods are correct?

(2) if any one is correct, is the answer equivalent to the previous answer?

3. Method 3

Since $x^4 + \frac{1}{x^4} = 10$, we have

$x^4 + 2 + \frac{1}{x^4} = 12$ and $x^4 - 2 + \frac{1}{x^4} = 8$

$\left(x^2 + \frac{1}{x^2}\right)^2 = 12$ and $\left(x^2 - \frac{1}{x^2}\right)^2 = 8$

$x^2 + \frac{1}{x^2} = \pm\sqrt{12}$ and $x^2 - \frac{1}{x^2} = \pm\sqrt{8}$

Adding, $2x^2 = \pm\sqrt{12} \pm \sqrt{8} \Rightarrow x^2 = \pm\sqrt{3} \pm \sqrt{2} \Rightarrow x = \pm\sqrt{\pm\sqrt{3} \pm \sqrt{2}}$

Subtracting, $\frac{2}{x^2} = \pm\sqrt{12} \mp \sqrt{8} \Rightarrow \frac{1}{x^2} = \pm\sqrt{3} \mp \sqrt{2} \Rightarrow \frac{1}{x} = \pm\sqrt{\pm\sqrt{3} \mp \sqrt{2}}$

$\therefore \quad x + \frac{1}{x} = \pm\sqrt{\pm\sqrt{3} \pm \sqrt{2}} \pm \sqrt{\pm\sqrt{3} \mp \sqrt{2}}$

4. Method 4

Since $x^4 + \frac{1}{x^4} = 10$, we have $x^8 - 10x^4 + 1 = 0$

We use quadratic equation formula and get $x^4 = 5 \pm \sqrt{24}$

$$x = \pm\sqrt[4]{5 \pm \sqrt{24}}$$

$$\therefore \quad x + \frac{1}{x} = \pm\sqrt[4]{5 \pm \sqrt{24}} \pm \frac{1}{\sqrt[4]{5 \pm \sqrt{24}}} = \pm\sqrt[4]{5 \pm \sqrt{24}} \pm \frac{\sqrt[4]{5 \mp \sqrt{24}}}{\sqrt[4]{(5 \pm \sqrt{24})(5 \mp \sqrt{24})}}$$

$$= \pm\sqrt[4]{5 \pm \sqrt{24}} \pm \sqrt[4]{5 \mp \sqrt{24}}$$